

Towards Decision Support in Reciprocation*

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Abstract

People often interact repeatedly: with relatives, through file sharing, in politics, etc. Many such interactions are reciprocal: reacting to the actions of the other. In order to facilitate decisions regarding reciprocal interactions, we analyze the development of reciprocation over time. To this end, we propose a model for such interactions that is simple enough to enable formal analysis, but is sufficient to predict how such interactions will evolve. Inspired by existing models of international interactions and arguments between spouses, we suggest a model with two reciprocating attitudes where an agent's action is a weighted combination of the others' last actions (reacting) and either i) her innate kindness, or ii) her own last action (inertia). We analyze a network of repeatedly interacting agents, each having one of these attitudes, and prove that their actions converge to specific limits. Convergence means that the interaction stabilizes, and the limits indicate the behavior after the stabilization. For two agents, we describe the interaction process and find the limit values. For a general connected network, we find these limit values if all the agents employ the second attitude, and show that the agents' actions then all become equal. In the other cases, we study the limit values using simulations. We discuss how these results predict the development of the interaction and can be used to help agents decide on their behavior.

Keywords reciprocal interaction, agents, action, repeated reciprocation, fixed, floating, behavior, network, convergence, Perron-Frobenius, convex combination

1 Introduction

Interaction is central in human behavior, e.g., at school, in file sharing, in business cooperation and political struggle. We aim at facilitating decision support

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for the interacting parties and for the outside observers. To this end, we want to predict interaction.

Instead of being economically rational, people tend to adopt other ways of behavior [30, 35], not necessarily maximizing some utility function. Furthermore, people tend to reciprocate, i.e., react on the past actions of others [15, 17, 21, 40]. Since reciprocation is ubiquitous, predicting it will allow predicting many real-life interactions and advising on how to improve them. Therefore, we need a model for reciprocating agents that is simple enough for analytical analysis and precise enough to predict such interactions. Understanding such a model would also help understanding how to improve personal and public good. This is also important for engineering computer systems that fit human intuition of reciprocity.

Extant models of (sometimes repeated) reciprocation can be classified as explaining existence or analyzing consequences. The following models consider the reasons for existence of reciprocal tendencies, often incorporating evolutionary arguments. The classical works of Axelrod [1, 2] considered discrete reciprocity and showed that it is rational for egoists, so that species evolve to reciprocate. Evolutionary explanation appears also in other places, such as [22, 39], or [5, Chapter 6], the latter also explicitly considering the psychological aspects of norm emergence. In [44], they consider pursuing fairness as a motivation for reciprocation. In [3] and [18], they considered engendering reciprocation by both the genetical kinship theory (helping relatives) and by the utility from cooperating when the same pair of agents interact multiple times. The famous work of Trivers [41] showed that sometimes reciprocity is rational, in much biological detail, and thus, people can evolve to reciprocate. Gintis [19, Chapter 11] considered discrete actions, discussing not only the rationally evolved tit-for-tat, but also reciprocity with no future interaction in sight, what he calls *strong reciprocity*. He modeled the development of strong reciprocity. Several possible reasons for strong reciprocity, such as a social part in the utility of the agents, expressing itself in emotions, were considered in [16]. Berg et al. [4] proved that people tend to reciprocate and considered possible motivations, such as evolutionary stability. Reciprocal behavior was axiomatically motivated in [37], assuming agents care not only for the outcomes, but also for strategies, thereby pushed to reciprocate.

On another research avenue, Given that reciprocal tendencies exist, the following works analyzed what ways it makes interactions develop. Some models analyzed reciprocal interactions by defining and analyzing a game where the utility function of rational agents directly depends on showing reciprocation, such as in [11, 14, 15, 29]. The importance of reward/punishment or of incomplete contracts for the flourishing of reciprocal individuals in the society was shown in [17].

To summarize, reciprocity is seen as an inborn quality [16, 41], which has probably been evolved from rationality of agents, as was shown by Axelrod [1]. As we have already said, understanding how a reciprocal interaction between agents with various reciprocal inclinations uncurls with time will help explain and predict the dynamics of reciprocal interaction, such as arms races and per-

sonal relations. This would also be in the spirit of the call to consider various repercussions of reciprocity from [27]. Since no analysis considers non-discrete lengthy interaction, caused by inborn reciprocation, (unlike, say, the discrete one from Axelrod [1, 2]), we model and study how reciprocity makes interaction evolve with time.

We represent actions by *weight*, where a bigger value means a more desirable contribution or, in the interpersonal context, investment in the relationship. We model reciprocity by two reciprocation attitudes, an action's weight being a convex combination between i) one's own kindness or ii) one's own last action, and the other's and neighborhood's last actions. The whole past should be considered, but we assume that the last actions represent the history enough, to facilitate analysis. Defining an action (or how much it changes) or a state by a linear combination of the other side's actions and own actions and qualities was also used to analyze arms race [13, 45] and spouses' interaction [20] (piecewise linear in this case). Attitude i) depending on the (fixed) kindness is called *fixed*, and ii) depending on one's own last action is called *floating*. Given this model, we study its behavioral repercussions.

There are several reminiscent but different models. The *floating* model resembles opinions that converge to a consensus [7, 26, 42, 12], while the *fixed* model resembles converging to a general equilibrium of opinions [6]. Of course, unlike the models of spreading opinions, we consider different actions on various neighbors, determined by direct reaction and a reaction to the whole neighborhood. Still, because of some technical reminiscence to some of our models, we do use those for one of our proofs. Another similar model is that of monotonic concession [33] and that of bargaining over dividing a pie [34]. The main difference is that in those models, the agents decide what to do, while in our case, they follow the reciprocation formula.

Example 1. *Consider n colleagues $1, 2, \dots, n$, who can help or harm each other. Let the possible actions be: giving bad work, showing much contempt, showing little contempt, supporting emotionally a little, supporting emotionally a lot, advising, and let their respective weight be a point in $[-1, -0.5)$, $[-0.5, -0.2)$, $[-0.2, 0)$, $(0, 0.4)$, $[0.4, 0.7)$, $[0.7, 1]$. Assume that each person knows what the other did to him last time. The social climate, meaning what the whole group did, also influences behavior. However, we may just concentrate on a single pair of even-tempered colleagues who reciprocate regardless the others.*

To understand and predict reciprocal behavior, we look at the limit of time approaching infinity, since this describes what actions will take place from some time on. We first consider two agents in Section 3, assuming their interaction is independent of other agents, or that the total influence of the others on the pair is negligible. This assumption allows for deeper a theoretical analysis of the interaction than in the general case. The values in the limit for two agents will be also implied by a general convergence result that is presented later, unless both agents are *fixed*. We still present them with the other results for two agents for the completeness of Section 3. Section 5 studies interaction of many agents, where the techniques we used for two agents are not applicable, and we

show exponentially fast convergence. Exponential convergence means a rapid stabilizing, and it explains acquiring personal behavioral styles, which is often seen in practice [32]. We find the limit when all the agents act synchronously and at most one has the *fixed* reciprocation attitude. Among other things, we prove that when at most one agent is *fixed*, the limits of the actions of all agents are the same, explaining formation of organizational subcultures, known in the literature [24]. We also find that only the kindness values of the *fixed* agents influence the limits of the various actions, thereby explaining that persistence (i.e., being faithful to one’s inner inclination) makes interaction go one’s own way, which is reflected in daily life in the recommendations to reject undesired requests by firmly repeating the reasons for rejection [8, Chapter 1] and [43, Chapter 8]. Other cases are simulated in Section 6. These results describe the interaction process and lay the foundation for further analysis of interaction.

The major contributions are proving convergence and finding its limits for at most one *fixed* agent or for two agents. These allow to explain the above mentioned phenomena and predict reciprocation. The predictions can assist in deciding whether a given interaction will be profitable, and in engineering more efficient multi-agent systems, fitting the reciprocal intuition of the users.

2 Modeling Reciprocation

2.1 Basic

Let $N = \{1, 2, \dots, n\}$ be $n \geq 2$ interacting agents. We assume that possible actions are described by an undirected interaction graph $G = (N, E)$, such that agent i acts on j and vice versa if and only if $(i, j) \in E$. Denote the degree of agent $i \in N$ in G by $d(i)$. This allows for various topologies, including heterogeneous ones, like those in [36]. To be able to mention directed edges, we shall treat this graph as a directed one, where for every $(i, j) \in E$, we have $(j, i) \in E$. Time is modeled by a set of discrete moments $t \in T \triangleq \{0, 1, 2, \dots\}$, defining a time slot whenever at least one agent acts. Agent i acts at times $T_i \triangleq \{t_{i,0} = 0, t_{i,1}, t_{i,2}, \dots\} \subseteq T$, and $\cup_{i \in N} T_i = T$. We assume that all agents act at $t = 0$, since otherwise we cannot sometimes consider the last action of another agent, which would force us to complicate the model and render it even harder for theoretical analysis. When all agents always act at the same times ($T_1 = T_2 = \dots = T_n = T$), we say they act *synchronously*.

For the sake of asymptotic analysis, we assume that each agent gets to act an infinite number of times; that is, T_i is infinite for every $i \in N$. Any real application will, of course, realize only a finite part of it, and infinity models the unboundedness of the process in time.

When (i, j) is in E , we denote the weight of an action by agent $i \in N$ on another agent $j \in N$ at moment t by $\text{act}_{i,j}(t): T_i \rightarrow \mathbb{R}$. We extend $\text{act}_{i,j}$ to T by assuming that at $t \in T \setminus T_i$, we have $\text{act}_{i,j}(t) = 0$. Since only the weight of an action is relevant, we usually write “action” while referring to its weight. For example, when interacting by file sharing, sending a valid piece of a file, nothing,

or a piece with a virus has a positive, zero, or a negative weight, respectively.

For $t \in T$, we define *the last action time* $s_i(t): T \rightarrow T_i$ of agent i as the largest $t' \in T_i$ that is at most t . Since $0 \in T_i$, this is well defined. The last action of agent i on (another) agent j is given by $x_{i,j}(t) \triangleq \text{act}_{i,j}(s_i(t))$. Thus, we have defined $x_{i,j}(t): T \rightarrow \mathbb{R}$, and we use mainly this concept $x_{i,j}$ in the paper. We denote the total received contribution from all the neighbors $N(i)$ at their last action times not later than t by $\text{got}_i(t): T \rightarrow \mathbb{R}$; formally, $\text{got}_i(t) \triangleq \sum_{j \in N(i)} x_{j,i}(t)$.

We now define two reciprocation attitudes, which define how an agent reciprocates. We need the following notions. The kindness of agent i is denoted by $k_i \in \mathbb{R}$; w.l.o.g., $k_n \geq \dots \geq k_2 \geq k_1$ throughout the paper. Kindness models inherent inclination to help others; in particular, it determines the first action of an agent, before others have acted. We model agent i 's inclination to mimic a neighboring agent's action and the actions of the whole neighborhood in G by reciprocation coefficients $r_i \in [0, 1]$ and $r'_i \in [0, 1]$ respectively, such that $r_i + r'_i \leq 1$. Here, r_i is the fraction of $x_{i,j}(t)$ that is determined by the last action of j upon i , and r'_i is the fraction that is determined by $\frac{1}{|N(i)|}$ th of the total contribution to i from all the neighbors at the last time.

2.2 Reciprocation

Intuitively, the *fixed* attitude depends on the agent's kindness at every action, while the *floating* one is loose, moving freely in the reciprocation process, and kindness directly influences such behavior only at $t = 0$. In both cases $x_{i,j}(0) \triangleq k_i$.

Definition 1. For the fixed reciprocation attitude, agent i 's reaction on the other agent j and on the neighborhood is determined by the agent's kindness weighted by $1 - r_i - r'_i$, by the other agent's action weighted by r_i and by the total action of the neighbors weighted by r'_i and divided over all the neighbors: That is, for $t \in T_i$, $\text{act}_{i,j}(t) = x_{i,j}(t) \triangleq$

$$(1 - r_i - r'_i) \cdot k_i + r_i \cdot x_{j,i}(t - 1) + r'_i \cdot \frac{\text{got}_i(t - 1)}{|N(i)|}.$$

Definition 2. In the floating reciprocation attitude, agent i 's action is a weighted average of her own last action, of that of the other agent j and of the total action of the neighbors divided over all the neighbors: To be precise, for $t \in T_i$, $\text{act}_{i,j}(t) = x_{i,j}(t) \triangleq$

$$(1 - r_i - r'_i) \cdot x_{i,j}(t - 1) + r_i \cdot x_{j,i}(t - 1) + r'_i \cdot \frac{\text{got}_i(t - 1)}{|N(i)|}.$$

The relations are (usually inhomogeneous) linear recurrences with constant coefficients. We could express the dependence $x_{i,j}(t)$ only on $x_{i,j}(t')$ with $t' < t$, but then the coefficients would not be constant, besides the case of two *fixed* agents. We are not aware of a method to use the general recurrence theory to improve our results.

2.3 Clarifications

Compared to the other models, our model takes reciprocal actions as given and looks at the process, while other models either consider how reciprocation originates, such as the evolutionary model of Axelrod [1], or take it as given and consider specific games, such as in [11, 14, 15, 29].

In Example 1, let (just here) $n = 3$ and the reciprocation coefficients be $r_1 = r_2 = 0.5, r'_1 = r'_2 = 0.3, r_3 = 0.8, r'_3 = 0.1$. Assume the kindness to be $k_1 = 0, k_2 = 0.5$ and $k_3 = 1$. Since this is a small group, all the colleagues may interact, so the graph is a clique¹. At $t = 0$, every agent's action on every other agent is equal to her kindness value, so agent 1 does nothing, agent 2 supports emotionally a lot, and 3 provides advice. If all agents act synchronously, meaning $T_1 = T_2 = T_3 = \{0, 1, \dots\}$, and all get carried away by the process, meaning that they forget the kindness in the sense of employing *floating* reciprocation, then, at $t = 1$ they act as follows: $x_{1,2}(1) = (1 - 0.5 - 0.3) \cdot 0 + 0.5 \cdot 0.5 + 0.3 \cdot \frac{0.5+1}{2} = 0.475$ (supports emotionally a lot), $x_{1,3}(1) = (1 - 0.5 - 0.3) \cdot 0 + 0.5 \cdot 1 + 0.3 \cdot \frac{0.5+1}{2} = 0.975$ (provides advice), $x_{2,1}(1) = (1 - 0.5 - 0.3) \cdot 0.5 + 0.5 \cdot 0 + 0.3 \cdot \frac{0+1}{2} = 0.25$ (supports emotionally a little), and so on.

Consider modeling tit for tat [2]:

Example 2. *In our model, the tit for tat with two options, - cooperate or defect, is easily modeled with $r_i = 1, k_i = 1$, meaning that the original action is cooperating (1) and the next action is the current action of the other player. Since we consider a mechanism, rather than a game, the agents will always cooperate. If one agent begins with cooperation ($k_1 = 1$) and the other one with defection ($k_2 = 0$), acting synchronously, then they will alternate.*

The notation is summarized in Table 1.

3 Pairwise Interaction

We now consider an interaction of two agents, 1 and 2, since this assumption allows proving much more than we will be able to in the general case. The model reduces to a pairwise interaction, when $r'_i = 0$ or when there are no neighbors besides the other agent in the considered pair. We assume both, w.l.o.g. Since agent 1 can only act on agent 2 and vice versa, we write $\text{act}_i(t)$ for $\text{act}_{i,j}(t)$, $x(t)$ for $x_{1,2}(t)$ and $y(t)$ for $x_{2,1}(t)$.

We analyze the case of both agents being *fixed*, then the case of the *floating*, and then the case where one is *fixed* and the other one is *floating*. To formally discuss the actions after the interaction has stabilized, we consider the limits (if exist)² $\lim_{p \rightarrow \infty} \text{act}_1(t_{1,p})$, and $\lim_{t \rightarrow \infty} x(t)$, for agent 1, and $\lim_{p \rightarrow \infty} \text{act}_2(t_{2,p})$ and $\lim_{t \rightarrow \infty} y(t)$ for agent 2. Since the sequence $\{x(t)\}$ is $\{\text{act}_1(t_{1,p})\}$ with finite repetitions, the limit $\lim_{t \rightarrow \infty} x(t)$ exists if and only if $\lim_{p \rightarrow \infty} \text{act}_1(t_{1,p})$ does. If

¹A clique is a fully connected graph.

²Agent i acts at the times in $T_i = \{t_{i,0} = 0, t_{i,1}, t_{i,2}, \dots\}$.

Term:	Meaning:
$\text{act}_{i,j}(t): T \rightarrow \mathbb{R}$	The action of i on another agent j at time t .
T_i	The time moments when agent i acts.
Synchronous	$T_1 = T_2 = \dots = T_n$.
$s_i(t): T \rightarrow T_i$	$\max \{t' \in T_i t' \leq t\}$.
$x_{i,j}(t): T \rightarrow \mathbb{R}$	$\text{act}_{i,j}(s_i(t))$.
$\text{got}_i(t): T \rightarrow \mathbb{R}$	$\sum_{j \in N(i)} x_{j,i}(t)$.
k_i	The kindness of agent i .
$r_i, r'_i \in [0, 1], r_i + r'_i \leq 1$	The reciprocation coefficients of agent i .
Agent i has the <i>fixed</i> reciprocation attitude, j is another agent	At moment $t \in T_i$, $x_{i,j}(t) \triangleq \begin{cases} (1 - r_i - r'_i) \cdot k_i + r_i \cdot x_{j,i}(t-1) \\ + r'_i \cdot \frac{\text{got}_i(t-1)}{ N(i) } & t > t_{i,0} \\ k_i & t = t_{i,0} = 0. \end{cases}$
Agent i has the <i>floating</i> reciprocation attitude, j is another agent	At moment $t \in T_i$, $x_{i,j}(t) \triangleq \begin{cases} (1 - r_i - r'_i) \cdot x_{i,j}(t-1) \\ + r_i \cdot x_{j,i}(t-1) + r'_i \cdot \frac{\text{got}_i(t-1)}{ N(i) } & t > t_{i,0} \\ k_i & t = t_{i,0} = 0. \end{cases}$

Table 1: The notation used throughout the paper.

they exist, they are equal; the same holds for $\lim_{t \rightarrow \infty} y(t)$ and $\lim_{p \rightarrow \infty} \text{act}_2(t_{2,p})$. Denote $L_x \triangleq \lim_{t \rightarrow \infty} x(t)$ and $L_y \triangleq \lim_{t \rightarrow \infty} y(t)$.

3.1 Fixed Reciprocation

Here we prove that both action sequences converge.

Theorem 1. *If the reciprocation coefficients are not both 1, which means $r_1 r_2 < 1$, then we have, for $i \in N$: $\lim_{p \rightarrow \infty} \text{act}_i(t_{i,p}) = \frac{(1-r_i)k_i + r_i(1-r_j)k_j}{1-r_i r_j}$.*

The assumption that not both reciprocation coefficients are 1 and the similar assumptions in the following theorems (such as $1 > r_i > 0$) mean that the agent neither ignores the other's action, nor does it copy the other's action. These are to be expected in real life. In Example 1, if agents 1 and 2 employ *fixed*

reciprocation, $r_1 = r_2 = 0.5, r'_1 = r'_2 = 0.0$ and $k_1 = 0, k_2 = 0.5$, then we obtain $L_x = \frac{0.5 \cdot (1-0.5) \cdot 0.5}{1-0.5 \cdot 0.5} = 1/6$ and $L_y = \frac{(1-0.5) \cdot 0.5}{1-0.5 \cdot 0.5} = 1/3$.

In order to prove this theorem, we first show that it is sufficient to analyze the synchronous case, i.e., $T_1 = T_2 = T$.

Lemma 1. *Consider a pair of interacting agents. Denote the action sequences in case both agents acted at the same time, (i.e., $T_1 = T_2 = T$), by $\{x'(t)\}_{t \in T}$ and $\{y'(t)\}_{t \in T}$, respectively. Then the action sequences³ $\{\text{act}_1(t_{1,p})\}_{p \in \mathbb{N}}, \{\text{act}_2(t_{2,p})\}_{p \in \mathbb{N}}$ are subsequences of $\{x'(t)\}_{t \in T}$ and $\{y'(t)\}_{t \in T}$, respectively.*

The proof follows from Definition 1 by induction. (Straightforward proofs in this paper have been replaced by their general ideas due to lack of space). Using this lemma, it is sufficient to further assume the synchronous case.

Lemma 2. *In the synchronous case, for every $t > 0 : x(2t-1) \geq x(2t+1)$, and for every $t \geq 0 : x(2t) \leq x(2t+2) \leq x(2t+1)$. By analogy, $\forall t > 0 : y(2t-1) \leq y(2t+1)$, and $\forall t \geq 0 : y(2t) \geq y(2t+2) \geq y(2t+1)$. All the inequations are strict if and only if $0 < r_1, r_2 < 1, k_2 > k_1$.⁴*

Since we also have $t \geq 0 : x(2t) \leq x(2t+1)$, we obtain $t > 0 : x(2t-1) \geq x(2t+1) \geq x(2t)$, and for every $t \geq 0 : x(2t) \leq x(2t+2) \leq x(2t+1)$. By analogy, $\forall t > 0 : y(2t-1) \leq y(2t+1) \leq y(2t)$, and $\forall t \geq 0 : y(2t) \geq y(2t+2) \geq y(2t+1)$. Intuitively, this means that the sequence $\{x(t)\}$ is alternating while its amplitude is getting smaller, and the same holds for the sequence $\{y(t)\}$, with another alternation direction. The intuitive reasons are that first, agent 1 increases her action, while 2 decreases it. Then, since 2 has decreased her action, so does 1, while since 1 has increased hers, so does 2. We now prove the lemma.

Proof. We employ induction. For $t = 0$, we need to show that $x(0) \leq x(2) \leq x(1)$ and $y(0) \geq y(2) \geq y(1)$. We know that $x(0) = k_1, x(1) = (1-r_1) \cdot k_1 + r_1 \cdot k_2$, and $y(0) = k_2, y(1) = (1-r_2) \cdot k_2 + r_2 \cdot k_1$. Since $y(1) \leq k_2$, we have $x(2) = (1-r_1) \cdot k_1 + r_1 \cdot y(1) \leq x(1)$. Since $y(1) \geq k_1$, we also have $x(2) = (1-r_1) \cdot k_1 + r_1 \cdot y(1) \geq x(0)$. The proof for y s is analogous.

For the induction step, for any $t > 0$, assume that the lemma holds for $t-1$, which means $x(2t-3) \geq x(2t-1)$ (for $t > 1$), $x(2t-2) \leq x(2t) \leq x(2t-1)$, and $y(2t-3) \leq y(2t-1)$ (for $t > 1$), $y(2t-2) \geq y(2t) \geq y(2t-1)$.

We now prove the lemma for t . By Definition 1, $x(2t-1) = (1-r_1)k_1 + r_1y(2t-2)$ and $x(2t+1) = (1-r_1)k_1 + r_1y(2t)$. Since $y(2t-2) \geq y(2t)$, we have $x(2t-1) \geq x(2t+1)$. By analogy, we can prove that $y(2t-1) \leq y(2t+1)$.

Also by definition, $x(2t) = (1-r_1)k_1 + r_1y(2t-1)$ and $x(2t+2) = (1-r_1)k_1 + r_1y(2t+1)$. Since $y(2t-1) \leq y(2t+1)$, we have $x(2t) \leq x(2t+2)$. By definition, $x(2t+1) = (1-r_1)k_1 + r_1y(2t)$. Since $y(2t) \geq y(2t-1)$, we conclude that $x(2t+1) \geq x(2t)$. By analogy, we prove that $y(2t+1) \leq y(2t)$. From

³Agent i acts at the times in $T_i = \{t_{i,0} = 0, t_{i,1}, t_{i,2}, \dots\}$.

⁴We always assume that $k_2 \geq k_1$.

this, we conclude that $x(2t+2) \leq x(2t+1)$, and we have shown that $x(2t) \leq x(2t+2) \leq x(2t+1)$. By analogy, we prove that $y(2t) \geq y(2t+2) \geq y(2t+1)$.

The equivalence of strictness in all the inequations to $0 < r_1, r_2 < 1, k_2 > k_1$ is proven by repeating the proof with strict inequalities in one direction, and by noticing that not having one of the conditions $0 < r_1, r_2 < 1, k_2 > k_1$ implies equality in at least one of the statements of the lemma. \square

With these results we now prove Theorem 1.

Proof. Using Lemma 1, we assume the synchronous case. We first prove convergence, and then find its limit. For each agent, Lemma 2 implies that the even actions form a monotone sequence, and so do the odd ones. Both sequences are bounded, which can be easily proven by induction, and therefore each one converges. The whole sequence converges if and only if both limits are the same. We now show that they are indeed the same for the sequence $\{x(t)\}$; the proof for $\{y(t)\}$ is analogous. $x(t+1) - x(t)$

$$\begin{aligned} &= (1-r_1)k_1 + r_1y(t) - (1-r_1)k_1 - r_1y(t-1) \\ &= r_1(y(t) - y(t-1)) = r_1r_2(x(t-1) - x(t-2)) = \dots \\ &= (r_1r_2)^{\lfloor t/2 \rfloor} \begin{cases} x(1) - x(0) & t = 2s, s \in \mathbb{N} \\ x(2) - x(1) & t = 2s+1, s \in \mathbb{N}. \end{cases} \end{aligned}$$

As $r_1r_2 < 1$, this difference goes to 0 as t goes to ∞ . Thus, $x(t)$ converges (and so does $y(t)$). To find the limits $L_x = \lim_{t \rightarrow \infty} x(t)$ and $L_y = \lim_{t \rightarrow \infty} y(t)$, notice that in the limit we have $(1-r_1)k_1 + r_1L_y = L_x$ and $(1-r_2)k_2 + r_2L_x = L_y$ with the unique solution: $L_x = \frac{(1-r_1)k_1 + r_1(1-r_2)k_2}{1-r_1r_2}$ and $L_y = \frac{(1-r_2)k_2 + r_2(1-r_1)k_1}{1-r_1r_2}$. \square

We see that $L_x \leq L_y$, which is intuitive, since the agents are always considering their kindness, so the kinder one acts with a bigger weight also in the limit. In the simulation of the actions over time in Figure 1, on the left, $y(t)$ is always larger than $x(t)$, and on the right, they alternate several times before $y(t)$ becomes larger.

3.2 Floating Reciprocation

If both agents have the *floating* reciprocation attitude, their action sequences converge to a common limit.

Theorem 2. *If $0 < r_1 + r_2 < 2$, then, as $t \rightarrow \infty$, $x(t)$ and $y(t)$ converge to the same limit. In the synchronous case ($T_1 = T_2 = T$), they both approach*

$$\frac{1}{2} \left(k_1 + k_2 + (k_2 - k_1) \frac{r_1 - r_2}{r_1 + r_2} \right) = \frac{r_2}{r_1 + r_2} k_1 + \frac{r_1}{r_1 + r_2} k_2.$$

The common limit of the actions is shown in Figure 2.

In Example 1, if agents 1 and 2 employ *fixed* reciprocation, $r_1 = r_2 = 0.5, r'_1 = r'_2 = 0.0$ and $k_1 = 0, k_2 = 0.5$, then we obtain $L_x = L_y = (1/2) \cdot 0 + (1/2) \cdot 0.5 = 0.25$.

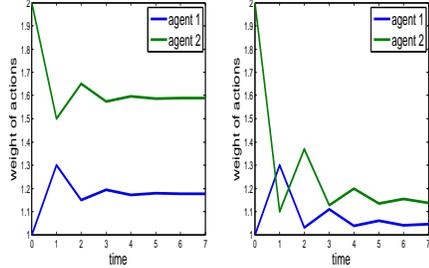


Figure 1: Simulation of actions for the synchronous case, with $r_1 + r_2 < 1$, $r_2 = 0.5$ on the left, and $r_1 + r_2 > 1$, $r_2 = 0.9$ on the right. This is a *fixed - fixed* reciprocation, with $k_1 = 1, k_2 = 2, r_1 = 0.3$. Each agent's oscillate, while converging to her own limit.

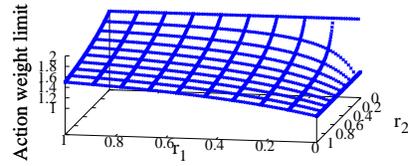


Figure 2: The common limit of the actions as a function of the reciprocity coefficients, for a *Floating - floating* reciprocation, $k_1 = 1, k_2 = 2$. Given r_2 , agent 1 receives most when $r_1 = 1$, and given r_1 , agent 2 receives most when $r_2 = 0$.

The idea of the proof is to show that $\{\{\min\{x(t), y(t)\}, \max\{x(t), y(t)\}\}_{t=1}^{\infty}$ is a nested sequence of segments, which lengths approach zero, and therefore, $\{x(t)\}$ and $\{y(t)\}$ converge to the same limit. Finding this limit stems from finding $\lim_{t \rightarrow \infty} (x(t) + y(t))$.

3.3 Fixed and Floating Reciprocation

Assume that agent 1 employs the *fixed* reciprocation attitude, while 2 acts by the *floating* reciprocation. We can show Theorem 3 using the following lemma.

Lemma 3. *If $r_2 > 0$ and $r_1 + r_2 \leq 1$, then, for every $t \geq t_{1,1} : x(t+1) \leq x(t)$, and for every $t \geq 0 : y(t+1) \leq y(t)$.*

The proof is by induction on t , using the definitions of reciprocation. With this lemma, we can prove the following.

Theorem 3. *If $r_2 > 0$ and $r_1 + r_2 \leq 1$, then, $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) = k_1$.*

Proof. We first prove that the convergence takes place, and then find its limit. For each agent, Lemma 3 implies that her actions are monotonically non-increasing. Since the actions are bounded below by k_1 , which can be easily proven by induction, they both converge.

To find the limits, notice that in the limit we have

$$(1 - r_1)k_1 + r_1L_y = L_x \tag{1}$$

$$(1 - r_2)L_y + r_2L_x = L_y. \tag{2}$$

From Eq. (2), we conclude that $L_x = L_y$, since $r_2 > 0$. Substituting this to Eq. (1) gives us $L_x = L_y = k_1$, since $r_2 > 0$ and $r_1 + r_2 \leq 1$ imply $r_1 < 1$. \square

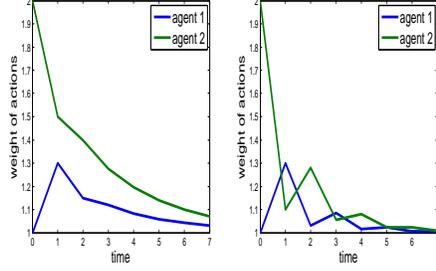


Figure 3: Simulation of actions for the synchronous case, with $r_1 + r_2 < 1$, $r_2 = 0.5$ on the left, and $r_1 + r_2 > 1$, $r_2 = 0.9$ on the right. This is a *fixed - floating* reciprocation, with $k_1 = 1, k_2 = 2, r_1 = 0.3$. In the left graph, agent 1's actions are smaller than those of 2; agent 1's actions decrease after $t = 1$, while those of agent 2 decrease all the time. The common limits' value fits the theorem's prediction.

The relation between the sequences of x s and y s is given by the following proposition (also covering the case $r_1 + r_2 \geq 1$).

Proposition 1. *If $r_1 + r_2 \leq 1$, then for every $t \geq 0 : y(t) \geq x(t)$. If $r_1 + r_2 \geq 1$, then $y(0) \geq x(0)$. For every $t > 0$ such that $t \in T_1 \cap T_2$, we have $y(t-1) \leq x(t-1) \Rightarrow y(t) \geq x(t)$. For any $t \in T_1 \setminus T_2$, we have $y(t) \geq x(t)$, and for any $t \in T_2 \setminus T_1$, we have $y(t-1) \geq x(t-1) \Rightarrow y(t) \geq x(t)$, and $y(t-1) \leq x(t-1) \Rightarrow y(t) \leq x(t)$.*

The proof employs induction on t .

We note that although we have not seen yet whether Theorem 3 holds for $r_1 + r_2 > 1$, we know that neither monotonicity (Lemma 3) nor $y(t)$ being always at least as large as $x(t)$ or the other way around holds in this case. As a counterexample for both of them, consider the case of $r_2 = 1, 0 < r_1 < 1, k_2 > k_1$. One can readily prove by induction that for all t we have $x(2t+1) > x(2t) = x(2t+2)$ and $y(2t) > y(2t-1) = y(2t+1)$, and thus both sequences are not monotonic. In addition, one can inductively prove that $x(2t+1) > y(2t+1), x(2t) < y(2t)$, and therefore no sequence is always larger than the other one.

Figure 3 shows how the actions evolve over time. The actions seem to converge also in the unproven case $r_1 + r_2 > 1$.

In the case of the mirroring assumption that agent 1 acts according to the *floating* reciprocation attitude, while 2 acts according to the *fixed* reciprocation, we can obtain similar results, which are omitted due to lack of space.

For all the considered cases, we have the following

Proposition 2. *If both L_x and L_y exist, then $L_x \leq L_y$.*

4 Alternating Case

We consider the interaction of two agents, 1 and 2. Some of the statements in the paper refer only to the synchronous case ($T_1 = T_2 = T$). All of them can be updated for the alternating case (T_1 contains precisely all the even times and T_2 contains zero and all the odd ones).

Theorem 2 can be extended as follows:

Theorem 4. *In the case where agents act alternately, which is when T_1 contains precisely all the even times and T_2 contains zero and all the odd ones, they both approach*

$$\begin{aligned} & \frac{1}{2} \left(k_1 + k_2 + \frac{(r_1 - r_2 - r_1 r_2)}{r_1 + r_2 - r_1 r_2} (k_2 - k_1) \right) \\ &= \frac{r_2}{r_1 + r_2 - r_1 r_2} k_1 + \frac{r_1 - r_1 r_2}{r_1 + r_2 - r_1 r_2} k_2. \end{aligned}$$

The idea of the proof is proving that $x(t) + y(t)$ approach $k_1 + k_2 + \frac{(r_1 - r_2 - r_1 r_2)}{r_1 + r_2 - r_1 r_2} (k_2 - k_1)$.

5 Multi-Agent Interaction

We now analyze the general interdependent interaction, when agents interact with many agents. To formally discuss the actions after the interaction has settled down, we consider the limits (if exist)⁵ $\lim_{p \rightarrow \infty} \text{act}_{i,j}(t_{1,p})$, and $\lim_{t \rightarrow \infty} x_{i,j}(t)$, for agents i and j . Since the sequence $\{x_{i,j}(t)\}$ is $\{\text{act}_{i,j}(t_{1,p})\}$ with finite repetitions, the limit $\lim_{p \rightarrow \infty} \text{act}_{i,j}(t_{1,p})$ exists if and only if $\lim_{t \rightarrow \infty} x_{i,j}(t)$ does. If they exist, they are equal. Denote $L_{i,j} \triangleq \lim_{t \rightarrow \infty} x_{i,j}(t)$.

We first provide general convergence results, and then we find the common limit for the case when at most one agent is *fixed* and synchronous in Theorem 6. We finally use simulations to analyze the limits in other cases. In this section, the ambivalent case of $r_i + r'_i = 1$ is taken to be *floating*.

First, we have convergence for the case of *floating* agents.

Proposition 3. *Consider a connected interaction graph, where all agents are floating and for every agent i , $r_i + r'_i < 1$. Then, for all pairs of agents $i \neq j$ such that $(i, j) \in E$, the limit $L_{i,j}$ exists; all these limits are equal to each other.*

Proof. Follows directly from [7, Theorem 2]. This article and similar articles on multiagent coordination [26, 42] prove convergence when all agents are *floating*. \square

We now show convergence, when some agents are *fixed*.

⁵Agent i acts at the times in $T_i = \{t_{i,0} = 0, t_{i,1}, t_{i,2}, \dots\}$.

Proposition 4. Consider a connected interaction graph, where for all agents i , $r'_i > 0$. Assume that at least one agent employs the fixed attitude and every agent acts at least once every q times, for a natural $q > 0$. Then, for all pairs of agents $i \neq j$ such that $(i, j) \in E$, the limit $L_{i,j}$ exists. The convergence is geometrically fast.

Proof. We express how each action depends on the actions in the previous time in matrix $A(t) \in \mathbb{R}_+^{|E| \times |E|}$, which, in the synchronous case, is defined as follows:

$$A(t)((i, j), (k, l)) \triangleq \begin{cases} (1 - r_i - r'_i) & \text{if } k = i, l = j; \\ r_i + r'_i \frac{1}{|N^+(i)|} & \text{if } k = j, l = i; \\ r'_i \frac{1}{|N^+(i)|} & \text{if } k \neq j, l = i; \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where the first line is missing for the *fixed* agents, since for them, own behavior does not matter. If, for each time $t \in T$, the column vector $\vec{p}(t) \in \mathbb{R}_+^{|E|}$ describes the actions at time t , in the sense that its (i, j) th coordinate contains $x_{i,j}(t)$ (for $(i, j) \in E$), then we have $\vec{p}(t+1) = A(t+1)\vec{p}(t) + \vec{k}'$, where \vec{k}' is the relevant kindness vector, formally defined as

$$k'(t)((i, j)) \triangleq \begin{cases} (1 - r_i - r'_i)k_i & \text{if } i \text{ is fixed;} \\ 0 & \text{otherwise.} \end{cases}$$

In a not necessarily synchronous case, only a subset of agents act at a given time t . For an acting agent i , every $A(t)((i, j), (k, l))$ is defined as in the synchronous case. For a non-acting agent i , we define

$$A(t)((i, j), (k, l)) \triangleq \begin{cases} 1 & \text{if } k = i, l = j; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

The kindness vector is defined as

$$k'(t)((i, j)) \triangleq \begin{cases} (1 - r_i - r'_i)k_i & \text{if } i \text{ is fixed and acting;} \\ 0 & \text{otherwise.} \end{cases}$$

By induction, we obtain $\vec{p}(t) = \prod_{t'=1}^t A(t')\vec{p}(0) + \sum_{\vec{k}' \in K} \left\{ \left(\sum_{l \in S_{\vec{k}'}(t)} \prod_{t'=l}^t A(t') \right) \vec{k}' \right\}$, where K is the set of all possible kindness vectors and $S_{\vec{k}'}(t)$ is a set of the appearance times of \vec{k}' , which are at most t .

We aim to show that $\vec{p}(t)$ converges. First, defining $r_i(M)$ to be the sum of the i th row of M , note [9, Eq. 3], namely

$$r_i(AB) = \sum_{j=1}^n \sum_{k=1}^n a_{i,j} b_{j,k} = \sum_{j=1}^n a_{i,j} r_j(B). \quad (5)$$

Since the sum of every row in any $A(t)$ is at most 1, we conclude that if $B \leq \beta C$, $C_{i,j} \equiv 1$, then also $A(t)B \leq \beta C$.

We now prove that an upper bound of the form βC on the entries of $\prod_{t=p}^q A(t)$ converges to zero geometrically. We have just shown that this bound never increases. First, $A(p) \leq C$, yielding the bound in the beginning. Now, let i be a *fixed* agent, and assume he acts at time t . Thus, each row in $A(t)$ which relates to the edges entering i sums to less than 1, and from Eq. (5) we gather that the upper bound on the appropriate rows in $A(t)B$ decreases relatively to the bound on B by some constant ratio. Since the graph is connected, for all agents i , $r'_i > 0$, and every agent acts every q times, we will have, after enough multiplications, that the bound on all the entries will have decreased by a constant ratio.

Every agent acts at least once every q times, so we gather that for some $q' > 0$, every q' times, the product of matrices becomes at most a given fraction of the product q' times before. This implies a geometric convergence of $\prod_{t'=1}^t A(t')$. As for $\sum_{l \in S_{\vec{k}}(t)} \prod_{t'=l}^t A(t')$, we have proven an exponential upper bound, thus $\sum_{l \in S_{\vec{k}}(t)} \prod_{t'=l}^t A(t') \leq \sum_{l \in S_{\vec{k}}(t)} \alpha^{\lceil \frac{t-l+1}{q'} \rceil} C \leq \sum_{l \in S_{\vec{k}}(t)} \alpha^{\frac{t-l}{q'}} C = \alpha^{\frac{t}{q'}} \left(\sum_{l \in S_{\vec{k}}(t)} \alpha^{\frac{-l}{q'}} \right) C \stackrel{\text{geom.}}{\leq} \text{seq.} \frac{\alpha^{\frac{t}{q'}} - 1}{\alpha^{\frac{1}{q'}} - 1} C$, proving a geometric convergence of the series $\sum_{l \in S_{\vec{k}}(t)} \prod_{t'=l}^t A(t')$. Therefore, $p(\vec{t})$ converges, and it does so geometrically fast. \square

As an immediate conclusion of this proposition, we can finally generalize Theorem 3 to the case $r_1 + r_2 > 1$ as follows.

Corollary 5. *Consider pairwise interaction, where one agent i employs fixed reciprocation and the other agent j employs the floating one, and every agent acts at least once every q times. Assume that $0 < r_i < 1$ and $r_j > 0$. Then, both limits exist and are equal to k_i . The convergence is geometrically fast.*

Proof. Proposition 4 implies geometrically fast convergence. We find the limits as in the proof of Theorem 3. \square

We now turn to finding the limit. We manage to do this only in the synchronous case, when all the agents are *floating* or all the *fixed* agents have the same kindness. For all reciprocation attitudes, the following theorem also provides an alternative proof of convergence in the synchronous case.

Theorem 6. *Given a connected interaction graph, consider the synchronous case where for all agents i , $r'_i > 0$. If there exists a cycle of an odd length in the graph (or at least one agent i employs floating reciprocation and has $r_i + r'_i < 1$), then, for all pairs of agents $i \neq j$ such that $(i, j) \in E$, the limit $L_{i,j}$ exists and it is a positive combination of all the kindness values of the agents who are fixed, if at least one agent is fixed, and of all the kindness values k_1, \dots, k_n , if all agents are floating. The convergence is geometrically fast. Moreover, if*

all agents employ floating reciprocation, then all these limits are equal to each other and it is a convex combination of the kindness values, namely

$$L = \frac{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \cdot k_i \right)}{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \right)}. \quad (6)$$

If, on the other hand, all the fixed agents have the same kindness k , then all these limits are equal to k . In any case, when not all the agents are floating, then changing only the kindness of the floating agents leaves all the limits as before (also follows from the limits being positive combinations of all the kindness values of the agents who are fixed).

Let us say several words about the assumptions. If all agents are *fixed*, we can prove that the actions are subsequences of the actions in the synchronous case (a straightforward generalization of Lemma 1.) Thus, the synchronous case represents all the cases in the limit, when all agents are *fixed*. The assumption of a cycle of an odd length virtually always holds, since three people influencing each other form such a cycle.

Proof. We first prove the case where all agents use *floating* reciprocation. We express how each action depends on the actions in the previous time in a matrix, and prove the theorem by applying the famous Perron–Frobenius theorem [38, Theorem 1.1, 1.2] to this matrix. We now define the dynamics matrix $A \in \mathbb{R}_+^{|E| \times |E|}$:

$$A((i, j), (k, l)) \triangleq \begin{cases} (1 - r_i - r'_i) & \text{if } k = i, l = j; \\ r_i + r'_i \frac{1}{|N^+(i)|} & \text{if } k = j, l = i; \\ r'_i \frac{1}{|N^+(i)|} & \text{if } k \neq j, l = i; \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

According to the definition of *floating* reciprocation, if for each time $t \in T$ the column vector $\vec{p}(t) \in \mathbb{R}_+^{|E|}$ describes the actions at time t , in the sense that its (i, j) th coordinate contains $x_{i,j}(t)$ (for $(i, j) \in E$), then $\vec{p}(t+1) = A\vec{p}(t)$. We then call $\vec{p}(t)$ an action vector. Initially, $\vec{p}_{(i,j)}(0) = k_i$.

Further, we shall need to use the Perron–Frobenius theorem for primitive matrices. We now prepare to use it, and first we show that A is primitive. First, A is irreducible since we can move from any $(i, j) \in E$ to any $(k, l) \in E$ as follows. We can move from an action to its reverse, since if $k = j, l = i$, then $A((i, j), (k, l)) = r_i + r'_i \frac{1}{|N^+(i)|} > 0$. We can also move from an action to another action by the same agent, since we can move to any action on the same agent and then to its reverse. To move to an action on the same agent, notice that if $l = i$, then $A((i, j), (k, l)) \geq r'_i \frac{1}{|N^+(i)|} > 0$. Now, we can move from any action (i, j) to any other action (k, l) by moving to the reverse action (j, i) (if $k = j, l = i$, we are done). Then, follow a path from j to k in graph G by moving to the appropriate action by an agent and then to the reverse, as many

times as needed till we are at the action (k, j) and finally to the action (k, l) . Thus, A is irreducible.

By definition, A is non-negative. A is aperiodic, since either at least one agent i has $r_i + r_i < 1$, and thus the diagonal contains non-zero elements, or there exists a cycle of an odd length in the interaction graph G . In the latter case, let the cycle be i_1, i_2, \dots, i_p for an odd p . Consider the following cycles on the index set of the matrix: $(i, j), (j, i), (i, j)$ for any $(i, j) \in E$ and $(i_2, i_1), (i_3, i_2), \dots, (i_p, i_{p-1}), (i_1, i_p), (i_2, i_1)$. Their lengths are 2 and p , respectively, which greatest common divisor is 1, implying aperiodicity. Being irreducible and aperiodic, A is primitive by [38, Theorem 1.4]. Since the sum of every row is 1, the spectral radius is 1.

According to the Perron–Frobenius theorem for primitive matrices [38, Theorem 1.1], the absolute values of all eigenvalues except one eigenvalue of 1 are strictly less than 1. The eigenvalue 1 has unique right and left eigenvectors, up to a constant factor. Both these eigenvectors are strictly positive. Therefore, [38, Theorem 1.2] implies that $\lim_{t \rightarrow \infty} A^t = \vec{1} \vec{v}'$, where \vec{v}' is the left eigenvector of the value 1, normalized such that $\vec{v}' \vec{1} = 1$, and the approach rate is geometric. Therefore, we obtain $\lim_{t \rightarrow \infty} \vec{p}(t) = \lim_{t \rightarrow \infty} A^t \vec{p}(0) = \vec{1} \vec{v}' \vec{p}(0) = \vec{1} \sum_{(i,j) \in E} v'((i,j)) k_i$. Thus, actions converge to $\vec{1}$ times $\sum_{(i,j) \in E} v'((i,j)) k_i$.

To find this limit, consider the vector v' defined by $v'((i,j)) = \frac{1}{r_i + r'_i}$. Substitution shows it is a left eigenvector of A . To normalize it such that $\vec{v}' \vec{1} = 1$, divide this vector by the sum of its coordinates, which is $\sum_{i \in N} \frac{d(i)}{r_i + r'_i}$, obtaining $v'((i,j)) = \frac{1}{\sum_{i \in N} \frac{d(i)}{r_i + r'_i}} \cdot \frac{1}{r_i + r'_i}$. Therefore, the common limit is $\frac{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \cdot k_i \right)}{\sum_{i \in N} \left(\frac{d(i)}{r_i + r'_i} \right)}$.

We now prove the case where at least one agent employs *fixed* reciprocation. We define the dynamics matrix A analogously to the previous case, besides that the first line from Eq. (7) is missing for the *fixed* agents, since for them, own behavior does not matter. In this case, we have $\vec{p}(t+1) = A \vec{p}(t) + \vec{k}'$, where \vec{k}' is the relevant kindness vector, formally defined as

$$k'((i,j)) \triangleq \begin{cases} (1 - r_i - r'_i) k_i & \text{if } i \text{ is } \textit{fixed}; \\ 0 & \text{otherwise.} \end{cases}$$

By induction, we obtain $\vec{p}(t) = A^t \vec{p}(0) + \left(\sum_{l=0}^{t-1} A^l \right) \vec{k}'$.

Analogically to the previous case, A is irreducible and non-negative. As shown above, A is aperiodic, and therefore, primitive. Since at least one agent employs *fixed* reciprocation, at least one row of A sums to less than 1, and therefore the spectral radius of A is strictly less than 1.

Now, the Perron–Frobenius implies that all the eigenvalues are strictly smaller than 1. Since we have $\lim_{t \rightarrow \infty} \vec{p}(t) = \lim_{t \rightarrow \infty} A^t \vec{p}(0) + \left(\lim_{t \rightarrow \infty} \sum_{l=0}^{t-1} A^l \right) \vec{k}'$, [38, Theorem 1.2] implies that this limits exist (the first part converges to zero, while the second one is a series of geometrically decreasing elements.) Since A is primitive, $\left(\lim_{t \rightarrow \infty} \sum_{l=0}^{t-1} A^l \right) > 0$.

When all the fixed agents have the same kindness k , we now find the limits. Taking the limits in the equality $\vec{p}(t+1) = A\vec{p}(t) + \vec{k}'$ yields $(I - A) \lim_{t \rightarrow \infty} \vec{p}(t) = \vec{k}'$. [38, Lemma B.1] implies that $I - A$ is invertible and therefore, if we guess a vector \vec{x} that fulfills $(I - A)\vec{x} = \vec{k}'$, it will be the limit. Since the vector with all actions equal to k satisfies this equation, we conclude that all the limits are equal to k . In any case, when there exists at least one *fixed* agent, changing only the kindness of the *floating* agents will not change the (unique) solution of $(I - A)\vec{x} = \vec{k}'$, and, therefore, will not change the limits. \square

Let us consider several examples of Eq. (6).

Example 3. *If the interaction graph is regular, meaning that all the degrees are equal to each other, we have $L = \frac{\sum_{i \in N} \left(\frac{k_i}{r_i + r'_i} \right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \right)}$. This holds for cliques, modeling small human collectives or groups of countries, and for cycles, modeling circular computer networks.*

Example 4. *For star networks, modeling networks of a supervisor of several people or entities, assume w.l.o.g. that agent 1 is the center, and we have $L = \frac{\frac{n-1}{r_1 + r'_1} \cdot k_1 + \sum_{i \in N \setminus \{1\}} \left(\frac{k_i}{r_i + r'_i} \right)}{\frac{n-1}{r_1 + r'_1} + \sum_{i \in N \setminus \{1\}} \left(\frac{1}{r_i + r'_i} \right)}$.*

An obvious conclusion of the theorem is that the *fixed* agents are, intuitively spoken, more important than the *floating* ones, at least their kindness is. We now conclude about the optimal reciprocation, which goes back to providing decision support.

Proposition 5. *If Eq. (6) holds, then agent i who wants to maximize the common value L , and who can choose either r_i or r'_i , in certain limits $[a, b]$, for $a > 0$, should choose either the smallest possible or the largest possible coefficient, as follows. We assume we choose r_i , but the same holds for r'_i with the obvious adjustments. She should set r_i to b , if $\sum_{j \in N \setminus \{i\}} \left(\frac{d(j)}{r_j + r'_j} \cdot k_j \right) - k_i \left(\sum_{j \in N \setminus \{i\}} \left(\frac{d(j)}{r_j + r'_j} \right) \right)$ is positive, to a , if that is negative, and to an arbitrary, if zero. When this expression is not zero, only these choices are optimal.*

The proof considers the sign of the derivative, and is omitted due to lack of space.

6 Simulations

We now answer some theoretically unanswered questions from Section 5 using MatLab simulations, running at least 100 synchronous rounds, to achieve practical convergence.

We first concentrate on the case of three agents who can influence each other, meaning that the interaction graph is a clique. We begin by corroborating the

already proven result that when at least one *fixed* agents exists, then the kindness of the *floating* agents does not influence the actions in the limit. Another proven thing we corroborate is that when exactly one *fixed* agent exists, then all the actions approach her kindness as time approaches infinity. When the actions are plotted as functions of time, we obtain graphs such as those in Figure 4. The left graph on that figure demonstrates, that exponential convergence may be quite slow, and this is a new observation we did not know from the theory. We also corroborate that the limiting values of the actions depend linearly on the kindness values of all the fixed agents, the proportionality coefficients being independent of the other kindness values. In order to reasonably cover the sampling space, all the above mentioned regularities have also been automatically checked for the combinations of kindness values of 1, 2, 3, 4, 5, over r_i and r'_i values of 0.1, 0.3, 0.5, 0.7, 0.9 and over all the relevant reciprocation attitudes. The checks were up to the absolute precision of 0.01.

We do not know the exact limits when there exist two or more *fixed* agents with distinct kindness values. We at least know that the dependencies on the kindness values are linear, but we lack theoretical knowledge about the dependencies of the limits of actions on the reciprocation coefficients, so we simulate the interaction for various reciprocation coefficients, obtaining graphs like those in Figure 5, and analogously for the dependency on r'_1 . Note that we can have both increasing and decreasing graphs in the same scenario, and also convex and concave graphs. The observed monotonicity was automatically verified for all the above mentioned combinations of parameters. This monotonicity means that if an agent wants to maximize the limit of the actions of some agent on some other agent, she can do this by choosing an extreme value of r_i or r'_i .

The next thing we study is a fourth agent, interacting with some of the other agents. We consider the limits of the actions as functions of the fourth agent's degree, but we found no regularity in these graphs; in particular, no monotonicity holds in the general case.

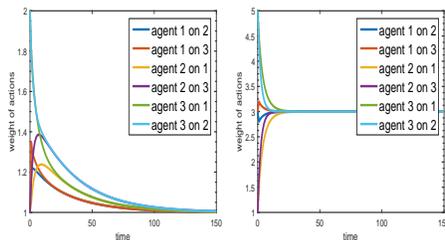


Figure 4: Simulation results for the synchronous case, with one *fixed* and two *floating* agents, for $r_1 = 0.1, r_2 = 0.1, r_3 = 0.1, r'_1 = 0.5, r'_2 = 0.1, r'_3 = 0.1$. In the left graph, $k_1 = 1, k_2 = 1, k_3 = 2$, while in the right one, $k_1 = 3, k_2 = 1, k_3 = 5$. The common limits, which are equal to the kindness of agent 1, fit the prediction of Theorem 6.

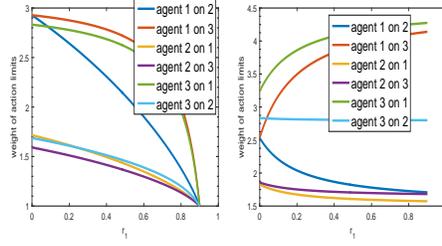


Figure 5: Simulation results for the synchronous case, where the limits of actions are plotted as functions of r_1 , for $r_2 = 0.1, r_3 = 0.6, r'_1 = 0.1, r'_2 = 0.4, r'_3 = 0.1, k_1 = 3, k_2 = 1, k_3 = 5$. In the left graph, agent 1 and 2 are the only *fixed* agents, while in the right one, 1 is the only *floating* agent. All the graphs exhibit monotonicity.

7 Additional Notes

When defining a reciprocating reaction, we used the last action of the other agent to model the opinion about the other agent. We can explicitly define the *opinion* of agent i about another agent j at time t , $\text{opin}_{i,j}: \mathbb{R}^{t+1} \rightarrow \mathbb{R}$, as $\text{opin}_{i,j}(t) \triangleq \text{act}_{j,i}(s_j(t))$, upon i . Then, we obtain that in the *fixed* reciprocation attitude for $t > 0$, $\text{act}_{i,j}(t) \triangleq (1 - r_i - r'_i) \cdot k_i + r_i \cdot \text{opin}_{i,j}(t-1) + r'_i \cdot \frac{\text{got}_i(t-1)}{|N(i)|}$. and in the *floating* reciprocation attitude for $t > 0$, $\text{act}_{i,j}(t) \triangleq (1 - r_i - r'_i) \cdot \text{act}_{i,j}(s_i(t-1)) + r_i \cdot \text{opin}_{i,j}(t-1) + r'_i \cdot \frac{\text{got}_i(t-1)}{|N(i)|}$.

Naturally, a more general definition of opinion is possible. To this end, we define the temporal distance in T_i , for an $i \in N$, which designates how many times agent i acted between two given times in T_i . Formally, for an $i \in N$ and two times $t_{i,l}, t_{i,m} \in T_i$, we define $d_{T_i}: T_i^2 \rightarrow \mathbb{R}_+$ by $d_{T_i}(t_{i,l}, t_{i,m}) \triangleq |l - m|$. Now, define the cumulative opinion of i about j at time t to be $\text{opin}_{i,j}(t) \triangleq \sum_{t' \in T_j, t' \leq t} \delta_i(d_{T_j}(t', s_j(t)) + 1) \cdot \text{act}_{j,i}(t')$, where $\delta_i(p): \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the discount function, expressing how much the passed time influences the importance of an action.

Our definition of opinion as $\text{opin}_{i,j}(t) = \text{act}_{j,i}(s_j(t))$ is a particular case of this model, where the discount function is $\delta_i(p) = \begin{cases} 1 & p = 1, \\ 0 & \text{otherwise.} \end{cases}$

8 Related Work

In addition to the direct motivation for our model, presented in Section 1, we were inspired by Trivers [41] (a psychologist), who describes a balance between

an inner quality (immutable kindness) and costs/benefits when determining an action. This idea of balancing the inner and the outer appears also in our model.

The idea of humans behaving according to a convex combination resembles another model, that of the altruistic extension, like [10, 23, 31], and Chapter iii.2 in [25]. In these papers, utility is often assumed being a convex combination, while we consider a mechanism of an action being a convex combination.

9 Conclusions and Future Work

In order to facilitate behavioral decisions regarding reciprocation, we need to predict what interaction a given setting will engender. To this end, we model two reciprocation attitudes where a reaction is a weighted combination of the action of the other player, the total action of the neighborhood and either one's own kindness or one's own last action. For a pairwise interaction, we show that actions converge, find the exact limits, and show that if you consider your kindness while reciprocating (*fixed*), then, asymptotically, your actions values get closer to your kindness than if you consider it only at the outset. For a general network, we prove convergence and find the common limit if all agents act synchronously and consider their last own action (*floating*), besides at most one agent. Dealing with the case when multiple agents consider their kindness (*fixed*) is mathematically hard, so we use simulations.

In Example 1 with the parameters from the end of Section 2, (all the agents employ *floating* reciprocation), Eq. (6) implies that all the actions approach $25/52$ in the limit, meaning that all the colleagues support each other emotionally a lot.

In addition to predicting the development of reciprocal interactions, our results explain why persistent agents have more influence on the interaction. An expression of the converged behavior is that while growing up, people acquire their own style of reciprocating with acquaintances [32]. In organizations, many styles are often very similar from person to person, forming organizational cultures [24].

We saw in theory and we know from everyday life that the reciprocation process may seem confusing, but the exponential convergence promises the confusion to be short. Actually, we can have a not so quick exponential convergence, such as observed in the left graph in Figure 4, but mostly, the process converges quickly. Another important conclusion is that employing *floating* reciprocation makes us achieve equality. In the synchronous case, to achieve a common limit it is also enough for all the *fixed* agents to have the same kindness. We also show that if all agents employ *floating* reciprocation and act synchronously, then the influence of an agent is proportional to her number of neighbors and inversely proportional to her tendency to reciprocate, that is, the stability. We prove that in the synchronous case, the limit is either a linear combination of the kindnesses of all the *fixed* agents or, if all the agents are *floating*, a linear combination of the kindnesses of all the agents. Thus, an agent's kindness influences nothing,

or it is a linear factor, thereby enabling a very eager agent to influence the limits arbitrarily, by having the *fixed* attitude and the appropriate kindness.

As we see in examples, real situations may require more complex modeling, motivating further research. For instance, modeling interactions with a known finite time horizon would be interesting. Since people may change while reciprocating, modeling changes in the reciprocity coefficients and/or reciprocation attitude is important. In addition, groups of colleagues and nations get and lose people, motivating modeling a dynamically changing set of reciprocating agents. Even with the same set of agents, the interaction graph may change as people move around. We study interaction processes where agents reciprocate with some given parameters, and show that maximizing L would require extreme values of reciprocation coefficients. To predict real situations better and to be able to give constructive advice about what parameters and attitudes of the agents are useful, we should define utility functions to the agents and consider the game where agents choose their own parameters before the interaction commences. This is hard, but people are able to change their behavior. Considering how to influence agents to change their behavior is also relevant. Though it seems extremely hard, it would be nice to consider our model in the light of a game theoretic model of an extensive form game, such as [14]. We used others' research, based on real data, as a basis for the model; actually evaluating the model on relevant data, like the arms race actions, may be enlightening. An agent could have different kindness values towards different agents, to represent her prejudgement. Another extension would be allowing the same action be perceived differently by various agents. A system of agents who have both a *fixed* and a *floating* component would be interesting to analyze.

Analytical and simulations analysis of reciprocation process allows estimating whether an interaction will be profitable to a given agent and lays the foundation for further modeling and analysis of reciprocation, in order to anticipate and improve the individual utilities and the social welfare.

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